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A Study of $H^3_{\text{sym}}(A, M)$

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1. DEFINITION OF H^2_{sym}, H^3_{sym} AND INTRODUCTION

Let k be a commutative ring with unit element. Let A be a k-algebra, M a k-module.

The Hochschild cohomology $H^{\cdot}(A, M)$ is defined as the cohomology of the complex:

$$C^n(A, M) = \text{Hom}(A^{\otimes n}, M),$$

where $A^{\otimes n} = A \otimes_k A \otimes \cdots \otimes_k A$ n times, and Hom means homomorphism as k-modules.

The coboundary map $\delta : C^n(A, M) \to C^{n+1}(A, M)$ is defined on $f \in C^n(A, M) = \text{Hom}(A^{\otimes n}, M)$ by

$$\begin{aligned}\delta f(a_1, \dots, a_{n+1}) = f(a_2, \dots, a_{n+1}) &+ \sum_{j=1}^{n} (-1)^j f(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\ &+ (-1)^{n+1} f(a_1, \dots, a_n).\end{aligned}$$

When A is a *commutative* algebra, one can define $H^2_{\text{sym}}(A, M)$ and $H^3_{\text{sym}}(A, M)$ as follows:

Let

$$C^2_{\text{sym}}(A, M) = \{f \in C^2(A, M) \mid f(a, b) = f(b, a), \quad \text{for all} \quad a, b \in A\}$$

$$C^3_{\text{sym}}(A, M) = \{f \in C^3(A, M) \mid f(a, b, c) - f(a, c, b) + f(c, a, b) = 0, \quad \text{for all } a, b, c \in A\}$$

One checks that

$$\delta C^1(A, M) \subseteq C^2_{\text{sym}}(A, M),$$

$$\delta C^2_{\text{sym}}(A, M) \subseteq C^3_{\text{sym}}(A, M),$$

and defining $Z^n(A, M)$ to be the kernel of $\delta : C^n(A, M) \to C^{n+1}(A, M)$, we set $Z^n_{\mathrm{sym}}(A, M) = Z^n(A, M) \cap C^n_{\mathrm{sym}}(A, M)$, for $n = 2, 3$. Then define

$$H^2_{\mathrm{sym}}(A, M) = Z^2_{\mathrm{sym}}(A, M)/\delta C^1(A, M),$$

$$H^3_{\mathrm{sym}}(A, M) = Z^3_{\mathrm{sym}}(A, M)/\delta C^2_{\mathrm{sym}}(A, M).$$

Let $k = Z$, the integers, and G be an Abelian group. Let $A = Z[G]$, the group algebra, and take M any Abelian group. Then Eilenberg and MacLane have shown in [4], p. 130, Theorem 26.3 that $H^3_{\mathrm{sym}}(A, M) = H^3_{\mathrm{sym}}(G, M) = 0$. In this paper we will show, by different methods, that $H^3_{\mathrm{sym}}(G, M) = 0$, if G is a finite abelian group. Also we will show that $H^3_{\mathrm{sym}}(A, M) = 0$, for the case when k is a field and Spec A is a finite group scheme over k. This is not really a generalization of the Eilenberg–MacLane result. The Cartier–Gabriel–Dieudonné theory gives us the fact that, if Spec A is a finite group scheme over k (definition later) and k is an algebraically closed field, then $A \cong k[G]$ for some finite abelian group G. The isomorphism $A \cong k[G]$ is of *algebras*. A also has a co-algebra structure which is probably different than the usual co-algebra structure on $k[G]$. As the choice of co-algebra structure for A is irrelevant for the group $H^3_{\mathrm{sym}}(A, M)$, we are back in the Eilenberg–MacLane situation. Moreover the reduction to the case of algebraically closed k is easy. We will give more details in Sections 2 and 3.

The result $H^3_{\mathrm{sym}}(A, M) = 0$ can be interpreted as a statement about symmetric cohomology of finite group schemes, which is useful to algebraic geometers (see Mumford–Oort [7], for example). This interpretation is carried out in Section 5.

2. $H^3_{\mathrm{sym}}(A, M)$ Studied as a Functor in M

Let k be a commutative ring with unit element. Let A be a commutative k-algebra which is free of finite type as a k-module.

Set $E = \mathrm{Hom}(A, K)$, the dual space, and let $m : A \otimes A \to A$ be the map: $m(a \otimes b) = ab$. The dual map $m^* : E \to E \otimes E$ makes E a co-algebra over k.

Also

$$C^n(A, M) = \mathrm{Hom}(A^{\otimes n}, M) = E^{\otimes n} \otimes M.$$

So when $M = k$; $C^n(A, k) = E^{\otimes n}$. The coboundary map (1.1): $E^{\otimes n} \to E^{\otimes n+1}$ is just the derivation δ on $\bigoplus_{n=0}^{\infty} E^{\otimes n}$ defined as follows;

For $f \in E$, $\delta f = 1 \otimes f + f \otimes 1 - m^* f$. For $f_1 \otimes \cdots \otimes f_n \in E^{\otimes n}$, $\delta(f_1 \otimes \cdots \otimes f_n) = \sum_{j=1}^{n} (-1)^{j+1} f_1 \otimes \cdots \delta f_j \otimes \cdots \otimes f_n$. And since $C^n(A, M) = E^{\otimes n} \otimes M$, the boundary map here is $\delta \otimes 1_M$, δ as above and $1_M : M \to M$

the identity map. The verification of the above statements is straightforward and left to the reader.

The symmetric chains $C^2_{\mathrm{sym}}(A, M)$, $C^3_{\mathrm{sym}}(A, M)$ can be computed via the following maps on $E^{\otimes 2}$, $E^{\otimes 3}$. Define $*$: $E^{\otimes 2} \to E^{\otimes 2}$ by $(f_1 \otimes f_2)^* = f_1 \otimes f_2 - f_2 \otimes f_1$ and $*$: $E^{\otimes 3} \to E^{\otimes 3}$ by

$$(f_1 \otimes f_2 \otimes f_3)^* = f_1 \otimes f_2 \otimes f_3 - f_1 \otimes f_3 \otimes f_2 + f_2 \otimes f_3 \otimes f_1 .$$

Let $E^{\otimes n}_{\mathrm{sym}}$ be the kernel of $* : E^{\otimes n} \to E^{\otimes n}$ for $n = 2, 3$.

PROPOSITION 2.1.

$$C^2_{\mathrm{sym}}(A, M) = E^{\otimes 2}_{\mathrm{sym}} \otimes M,$$

$$C^3_{\mathrm{sym}}(A, M) = E^{\otimes 3}_{\mathrm{sym}} \otimes M.$$

Proof. We are of course identifying $C^3(A, M)$ with $E^{\otimes 3} \otimes M$. Since E is free over k, it is enough to show that $C^3_{\mathrm{sym}}(A, M)$ is the kernel of $* \otimes 1$ on $E^{\otimes 3} \otimes M$. So take $f = f_1 \otimes f_2 \otimes f_3 \otimes m \in E^{\otimes 3} \otimes M = \mathrm{Hom}(A^{\otimes 3}, k) \otimes M$. Then

$$\begin{aligned} f(a, b, c) &- f(a, c, b) + f(c, a, b) \\ &= [f_1(a) f_2(b) f_3(c) - f_1(a) f_2(c) f_3(b) + f_1(c) f_2(a) f_3(b)]\, m \\ &= [(f_1 \otimes f_2 \otimes f_3 - f_1 \otimes f_3 \otimes f_2 + f_2 \otimes f_3 \otimes f_1) \otimes m](a, b, c), \end{aligned}$$

which implies $f(a, b, c) - f(a, c, b) + f(c, a, b) = [(* \otimes 1) f](a, b, c)$ and so $C^3_{\mathrm{sym}}(A, M)$ is the kernel of $* \otimes 1$.

The following will be useful in computations in Section 3.

PROPOSITION 2.2. *If $g \in E^{\otimes 2}_{\mathrm{sym}}$, $f \in E$, then*

$$(f \otimes g)^* = (g \otimes f)^* = g \otimes f.$$

Proof. g can be expressed $g = \sum_i a_i h_i \otimes h_i$ for $h_i \in E$, $a_i \in k$. So it is enough to check that $(f \otimes h \otimes h)^* = (h \otimes h \otimes f)^* = h \otimes h \otimes f$ for h, $f \in E$. This is easy to do using the definition of $*$.

THEOREM 2.1. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of k-modules.*

Then the induced sequence

$$\begin{aligned} 0 &\to H^1(A, M') \to H^1(A, M) \to H^1(A, M'') \\ &\to H^2_{\mathrm{sym}}(A, M') \to H^2_{\mathrm{sym}}(A, M) \to H^2_{\mathrm{sym}}(A, M'') \\ &\to H^3_{\mathrm{sym}}(A, M') \to H^3_{\mathrm{sym}}(A, M) \to H^3_{\mathrm{sym}}(A, M'') \end{aligned}$$

is exact.

Proof. We have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E \otimes M' & \longrightarrow & E \otimes M & \longrightarrow & E \otimes M'' & \to & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \to & E_{\mathrm{sym}}^{\otimes 2} \otimes M' & \to & E_{\mathrm{sym}}^{\otimes 2} \otimes M & \to & E_{\mathrm{sym}}^{\otimes 2} \otimes M'' & \to & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \to & E_{\mathrm{sym}}^{\otimes 3} \otimes M' & \to & E_{\mathrm{sym}}^{\otimes 3} \otimes M & \to & E_{\mathrm{sym}}^{\otimes 3} \otimes M'' & \to & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & &
\end{array}$$

The rows are exact since E is free over k and so the Theorem follows.

COROLLARY 2.2. *Let G be a finite Abelian group, let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of Abelian groups. Then $0 \to H^3_{\mathrm{sym}}(G, M') \to H^3_{\mathrm{sym}}(G, M) \to H^3_{\mathrm{sym}}(G, M'')$ is exact.*

Proof. $\quad H^1(G, M) \cong \mathrm{Hom}(G, M), \quad$ Hom as groups,

$H^2_{\mathrm{sym}}(G, M) \cong \mathrm{Ext}(G, M), \quad$ Ext of abelian groups,

and the first two lines of the exact sequence in Theorem 2.1 become

$$\begin{aligned} 0 \to \mathrm{Hom}(G, M') &\to \mathrm{Hom}(G, M) \to \mathrm{Hom}(G, M'') \\ &\to \mathrm{Ext}(G, M') \to \mathrm{Ext}(G, M) \to \mathrm{Ext}(G, M''); \end{aligned}$$

but it is well known that $\mathrm{Ext}(G, M) \to \mathrm{Ext}(G, M'')$ is surjective and the corollary follows.

Remark. From this corollary, it is easy to see that in order to show $H^3_{\mathrm{sym}}(G, M) = 0$, for all Abelian groups M, it is sufficient to show $H^3_{\mathrm{sym}}(G, M) = 0$ when M is divisible. This is what will be done in Section 4.

3. $H^3_{\mathrm{sym}}(A, M) = 0$ WHEN Spec A IS A FINITE GROUP SCHEME OVER A FIELD K

DEFINITION. Let k be a field. Let A be a commutative k-algebra via $\sigma : k \to A$, with multiplication $m : A \otimes A \to A$. Denote by 1 the identity map: $1a = a$ for all $a \in A$. Then A together with homomorphisms

$$\begin{aligned} \Delta &: A \to A \otimes A, \\ \epsilon &: A \to k, \\ s &: A \to A, \end{aligned}$$

is a group scheme over k if the following axioms hold.

(1) Associativity: $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$.

(2) Unit element: $(\epsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \epsilon) \circ \Delta$ (identifying $A \otimes_k k$ with A).

(3) inverses: $m \circ (1 \otimes s) \circ \Delta = \sigma \circ \epsilon = m \circ (s \otimes 1) \circ \Delta$.

(4) commutativity: If $T : A \otimes A \to A \otimes A$ is defined by $T(a \otimes b) = b \otimes a$, then $T \circ \Delta = \Delta$.

The A is a finite group scheme if, in addition to the above, A is finite dimensional as a k-module.

We now wish to compute $H^3_{\mathrm{sym}}(A, M)$ for M a k-module. We set $E = \mathrm{Hom}(A, k)$, the dual space.

PROPOSITION 3.1. $H^3_{\mathrm{sym}}(A, M) \cong H^3_{\mathrm{sym}}(A, k) \otimes M$.

Refer to Proposition 2.1 and use the universal coefficient theorem [6] p. 171, which gives the desired result since k is a field.

Thus it is enough to show $H^3_{\mathrm{sym}}(A, k) = 0$. Now let $\bar{k}$ be an algebraic closure of k. Let $\bar{A} = A \otimes_k \bar{k}$.

PROPOSITION 3.2. $H^3_{\mathrm{sym}}(\bar{A}, \bar{k}) \cong H^3_{\mathrm{sym}}(A, k) \otimes \bar{k}$.

Proof. Let $\bar{E} = E \otimes_k \bar{k} \cong \mathrm{Hom}(A, k) \otimes \bar{k}$, so $\bar{E}^{\otimes 3} \cong E^{\otimes 3} \otimes \bar{k}$ and thus $\bar{E}^{\otimes 3}_{\mathrm{sym}}$ is the kernel of $* \otimes 1 : E^{\otimes 3} \otimes \bar{k} \to E^{\otimes 3} \otimes \bar{k}$. So $\bar{E}^{\otimes 3}_{\mathrm{sym}}$ can be identified with $E^{\otimes 3}_{\mathrm{sym}} \otimes \bar{k}$ and similarly $\bar{E}^{\otimes 2}_{\mathrm{sym}}$ can be identified with $E^{\otimes 2}_{\mathrm{sym}} \otimes \bar{k}$. Then the universal coefficient theorem finishes the proof.

Notice that if Spec A is a finite group scheme over k, Spec $\bar{A}$ is a finite group scheme over $\bar{k}$, ($\bar{\Delta} = \Delta \otimes 1$ etc.) We are now reduced to the case where Spec A is a finite group scheme over an algebraically closed field k and we want to show $H^3_{\mathrm{sym}}(A, k) = 0$.

We first sketch how we can obtain a description of the graded ring $H^{\cdot}(A, k)$. This is carried out in detail in [3] so here only an outline is given.

The theory of Gabriel (expounded for instance in Manin [5], §3) combined with the theorem of Dieudonné–Cartier [2], VII b, p. 152, implies:

If A is a finite group scheme over an algebraically closed field k, of characteristic $p \neq 0$ then

(1) $A \cong A_{\mathrm{loc}} \otimes A_{\mathrm{et}}$.

(2) A_{et} is isomorphic as an algebra to $k[G_1]$ for some finite Abelian group G_1 whose order is relatively prime to p.

(3) A_{loc} is isomorphic as an algebra to $k[G_2]$, G_2 is a finite Abelian group whose order is a power of p.

It follows that $A \cong k[G_1 \times G_2] = k[G]$ as an algebra, where $G = G_1 \times G_2$, is a finite Abelian group. The $k[G]$ has a coalgebra structure but it is probably different from the given one on A.

Since A is a co-algebra, $H^{\cdot}(A, k)$ has the structure of a graded ring. It can be seen that the multiplication is induced by the following map on the chain level:

$$C^n(A, k) \otimes C^m(A, k) \to C^{n+m}(A, k),$$

which is $E^{\otimes n} \otimes E^{\otimes m} \to E^{\otimes n+m}$ and is given by

$$(f_1 \otimes \cdots \otimes f_n) \otimes (g_1 \otimes \cdots \otimes g_m) \to f_1 \otimes \cdots \otimes f_n \otimes g_1 \otimes \cdots g_n .$$

Denote the induced multiplication on $H^{\cdot}(A, k)$ by $\cup$. Then if $f \in H^n(A, k)$ and $g \in H^m(A, k)$ we get $f \cup g = (-1)^{mn} g \cup f$. Now as MacLane points out [*6*], p. 232, and we show directly in [*3*], the cup product is independent of the co-algebra structure on A (but there must be one). Thus we may as well assume that $A = k[G]$, and use the standard resolutions for cyclic abelian groups [*1*], p. 250 and the Künneth theorem to obtain the following theorem.

THEOREM 3.1. *If the characteristic of the field k is not* 2 *and* Spec A *is a finite group scheme over k, then the graded ring $H^{\cdot}(A, k)$ is freely generated as a commutative graded ring by $H^1(A, k)$ and $H^2_{\mathrm{sym}}(A, k)$. Freely generated means that the only relations are $f \cup g = (-1)^{mn} g \cup f$ if $f \in H^m$, $g \in H^n$.*

If the characteristic of k is 2, *then there are subgroups V_1 and V_2 of $H^1(A, k)$ and a subgroup V_3 of $H^2_{\mathrm{sym}}(A, k)$ such that V_1, V_2, V_3 generate $H^{\cdot}(A, k)$ as a commutative graded algebra subject to the added relations: $f^2 = 0$, if f is in V_1.*

This theorem enables us to directly compute $H^3_{\mathrm{sym}}(A, k)$. We will carry out the calculations when the characteristic p of the field k is not 2. The calculations for $p = 2$ are similar but one has to take into account the structure of $H^3(A, k)$ given in Theorem 3.1. If the characteristic of k is zero, then $H^n(A, k) = 0$, for $n > 0$, and we can conclude $H^3_{\mathrm{sym}}(A, k) = 0$ in this case as soon as we show $H^3_{\mathrm{sym}}(A, k)$ is a subgroup of $H^3(A, k)$.

Recall the following situation: We have k algebraically closed. Spec A is a finite group scheme over k, $E = \mathrm{Hom}(A, k)$, the dual space. Also $C^n(A, k) = \mathrm{Hom}(A^{\otimes n}, k) = E^{\otimes n}$.

Since $H^1(A, k)$ is a subgroup of $C^1(A, k) = E$, we can choose a basis $\{f_i\}_{i \in I'}$, for $H^1(A, k)$ and extend to a basis $\{f_i\}_{i \in I}$ for E over k, with index sets $I' \subseteq I$.

Note: $\delta f_i = 0$, for all i in I', and $\{\delta f_i\}_{i \in I - I'}$ is linearly independent in $E^{\otimes 2}$. We can also assume I' is ordered.

Choose a set $\{g_j\}_{j \in J}$ in $E^{\otimes 2}_{\mathrm{sym}}$ such that

(1) $\delta g_j = 0$, for all $j \in J$,

(2) If cls g_j denotes the image of g_j in $H^2_{\mathrm{sym}}(A, k)$, we have $\{\mathrm{cls}\, g_j\}_{j \in J}$ is a basis for $H^2_{\mathrm{sym}}(A, k)$.

Then by Theorem 3.1, we conclude:

$$\{\mathrm{cls}(g_j \otimes f_i\}_{(j,i)\in J\times I'} \quad \text{together with}$$
$$\{\mathrm{cls}(f_u \otimes f_v \otimes f_w)\}_{u<v<w\in I'}$$

is a basis for $H^3(A, k)$.

Thus, if $f \in Z^3_{\mathrm{sym}}(A, k)$, we have

$$f = \sum_{(i,j)\in I'\times J} a_{ij} g_j \otimes f_i + \sum_{u<v<w\in I'} b_{uvw} f_u \otimes f_v \otimes f_w + \delta h,$$

for some $h \in E^{\otimes 2}$. Also $f^* = 0$. We must show that under these conditions, $f = \delta h'$ for some h' in $E^{\otimes 2}_{\mathrm{sym}}$.

First we show $H^3_{\mathrm{sym}}(A, k)$ is a subgroup of $H^3(A, k)$.

LEMMA 3.1. *If $h \in E^{\otimes 2}$ and $(\delta h)^* = 0$, then $\delta h = \delta h'$, for some h' in $E^{\otimes 2}_{\mathrm{sym}}$.*

Proof. Using the basis $\{f_i \otimes f_j\}_{(i,j)\in I\times I}$ for $E \otimes E$, write

$$h = \sum_{(i,j)\in I\times I} c_{ij} f_i \otimes f_j .$$

One computes $\delta h = \sum c_{ij}(\delta f_i \otimes f_j - f_i \otimes \delta f_j)$ and so using Proposition 2.2

$$(\delta h)^* = \sum_{(i,j)\in I\times I} (c_{ij} - c_{ji})(\delta f_i \otimes f_j).$$

For fixed j we then get $\sum_i (c_{ij} - c_{ji})\, \delta f_i = 0$. So for fixed i, j; $c_{ij} \neq c_{ji}$ implies $\delta f_i = 0$ (by symmetry $\delta f_j = 0$). Let $U = \{(i, j) \mid c_{ij} = c_{ji}\}$. Then if

$$h' = \sum_{(i,j)\in U} c_{ij} f_i \otimes f_j ,$$

we get

$$\delta h' = \sum_{(i,j)\in U} c_{ij}(\delta f_i \otimes f_j - f_i \otimes \delta f_j) = \delta h,$$

since

$$\delta h - \delta h' = \sum_{(i,j)\notin U} c_{ij}(\delta f_i \otimes f_j - f_i \otimes \delta f_j) = 0.$$

COROLLARY 3.1. *If the characteristic of k is zero, then $H^3_{\mathrm{sym}}(A, k) = 0$.*

Proof. $H^3_{\mathrm{sym}}(A, k)$ is a subgroup of $H^3(A, k)$ which is zero.

Next we want to show that if $f \in Z^3_{\mathrm{sym}}$, then $f = \delta h$, for some $h \in E^{\otimes 2}$. We have

$$f = \sum_{(i,j)\in I'\times J} a_{ij} g_j \otimes f_i + \sum_{u<v<w\in I'} b_{uvw} f_u \otimes f_v \otimes f_w + \delta h.$$

LEMMA 3.2.

$$f^* = \sum_{(i,j)\in I'\times J} a_{ij} g_j \otimes f_i + \sum_{u<v<w\in I'} 3 f_u \otimes f_v \otimes f_w + \delta h_1 + (\delta h)^*,$$

for some $h_1 \in E^{\otimes 2}$.

Proof. First $(g_j \otimes f_i)^* = g_j \otimes f_i$ by Proposition 2.2. Second

$$(f_u \otimes f_v \otimes f_w)^* = f_u \otimes f_v \otimes f_w - f_u \otimes f_w \otimes f_v + f_v \otimes f_w \otimes f_u .$$

Taking classes in $H^3(A, k)$ and denoting class f_u by f_u, we get

$$\operatorname{cls}(f_u \otimes f_v \otimes f_w)^* = f_u \cup f_v \cup f_w - f_u \cup f_w \cup f_v + f_v \cup f_w \cup f_u = 3 f_u \cup f_v \cup f_w$$

by the description of $\cup$.

So $(f_u \otimes f_v \otimes f_w)^* = 3 f_u \otimes f_v \otimes f_w + \delta h_{uvw}$, for some $h_{uvw} \in E^{\otimes 2}$.

As a corollary of Lemma 3.2 we see that since $f^* = 0$, $0 = \delta f^* = \delta(\delta h)^*$.

LEMMA 3.3. *If* $h \in E^{\otimes 2}$ *and* $\delta((\delta h)^*) = 0$, *then* $(\delta h)^* = \delta h''$ *for some* $h'' \in E^{\otimes 2}$.

Proof. Write $h = \sum_{(i,j)\in I\times I} c_{ij} f_i \otimes f_j$. Then as before

$$(\delta h)^* = \sum_{(i,j)\in I\times I} (c_{ij} - c_{ji})(\delta f_i \otimes f_j).$$

Now

$$0 = \delta((\delta h)^*) = \sum_{(i,j)\in I\times I} (c_{ij} - c_{ji})(\delta f_i \otimes \delta f_j).$$

But $\{\delta f_i\}_{i\in I-I'}$ is linearly independent in $E^{\otimes 2}$. So, if $j \in I - I'$, $\sum_{i\in I-I'} (c_{ij} - c_{ji})\, \delta f_i = 0$. Thus

$$(\delta h)^* = \sum_{(i,j)\in I\times I'} (c_{ij} - c_{ji})(\delta f_i \otimes f_j),$$

and if we let

$$h'' = \sum_{(i,j)\in I\times I'} (c_{ij} - c_{ji})(f_i \otimes f_j),$$

we get $(\delta h)^* = \delta h''$.

Applying Lemma 3.3 to the expression for f^* in Lemma 3.2 we obtain

$$0 = f^* = \sum_{(i,j)\in I'\times J} a_{ij} g_j \otimes f_i + \sum_{u<v<w\in I'} 3 b_{uvw} f_u \otimes f_v \otimes f_w + \delta h_1 + \delta h''.$$

Taking images in $H^3(A, k)$ implies $a_{ij} = 0$, for all $(i,j) \in I' \times J$, and all $b_{uvw} = 0$, if the characteristic p of k is not 3. If $p = 3$ we need the following lemma.

LEMMA 3.4. *If* $p = 3$ *and* $\sum_{u<v<w\in I'} b_{uvw}(f_u \otimes f_v \otimes f_w)^* = (\delta h)^*$, *for some* $h \in E^{\otimes 2}$; *then* $(\delta h)^* = 0$.

Proof. This is a rather nasty calculation and goes as follows: Write

$$h = \sum_{(i,j)\in I\times I} c_{ij} f_i \otimes f_j .$$

Then

$$(\delta h)^* = \sum_{(i,j)\in I} (c_{ij} - c_{ji})\, \delta f_i \otimes f_j .$$

Set

$$h_j = \sum_i (c_{ij} - c_{ji}) f_i$$

Then

$$(\delta h)^* = \sum_{j\in I} (\delta h_j \otimes f_j);$$

But

$$\Big(\sum_{u<v<w} b_{uvw} f_u \otimes f_v \otimes f_w\Big)^*$$
$$= \sum_{u<v<w} b_{uvw}(f_u \otimes f_v \otimes f_w - f_u \otimes f_w \otimes f_v + f_v \otimes f_w \otimes f_u).$$

Fix $l \in I$, and we see

$$\delta h_l = \sum_{w=l} b_{uvw} f_u \otimes f_v - \sum_{v=l} b_{uvw} f_u \otimes f_w + \sum_{u=l} b_{uvw} f_v \otimes f_w .$$

Take images in $H^2(A, k)$ and we get

$$0 = \sum_{w=l} b_{uvw} f_u \cup f_v - \sum_{v=l} b_{uvw} f_u \cup f_w + \sum_{u=l} b_{uvw} f_v \cup f_w .$$

Suppose some $b_{uvw} \neq 0$, then in the above equation we must have cancellation somewhere. For example $b_{uvl} f_u \cup f_v = -b_{u'lw'} f_{u'} \cup f_{w'}$, and since $u < v$, $u' < w'$, this implies $u = u'$, $v = w'$. Then $v < l < w' = v$, a contradiction. All other possible cancellations similarly lead to contradictions.

THEOREM 3.2. *If k is a field,* Spec A *a finite group scheme over k, and M is a k-module. Then*

$$H^3_{\mathrm{sym}}(A, M) = 0.$$

Proof. We have reduced this to showing $H^3_{\mathrm{sym}}(A, k) = 0$, when k is algebraically closed. Assuming k algebraically closed, $f \in E^{\otimes 3}$, $f^* = 0$, $\delta f = 0$; Lemmas 3.3 and 3.4 show there exists h in $E^{\otimes 2}$, with $f^* = (\delta h)^*$. But $f^* = 0$ and so Lemma 3.1 implies there exists h' in $E^{\otimes 2}_{\mathrm{sym}}$, such that $f = \delta h'$. This completes the proof of Theorem 3.2.

4. $H^3_{\rm sym}(G, M) = 0$

THEOREM 4.1. *If G is a finite Abelian group, M any Abelian group, then* $H^3_{\rm sym}(G, M) = 0$.

Proof. By Theorem 1.1 we are reduced to considering the case $M = I$, a divisible Abelian group.

Let p be a prime and let I_p be the kernel of multiplication by $p : I \to I$. The sequence: $0 \to I_p \to I \xrightarrow{p} I \to 0$ is exact since I is divisible.

By corollary 2.2 we have

$$0 \to H^3_{\rm sym}(G, I_p) \to H^3_{\rm sym}(G, I) \xrightarrow{p} H^3_{\rm sym}(G, I) \qquad \text{is exact,}$$

and the map denoted by p is clearly multiplication by p. I_p is a Z/pZ module and

$$H^3_{\rm sym}(G, I_p) = H^3_{\rm sym}(Z/pZ[G], I_p) = 0,$$

by Theorem 3.2. This implies $H^3_{\rm sym}(G, I)$ has no p-torsion, and since p is arbitrary, we see that $H^3_{\rm sym}(G, I)$ has no torsion.

Now let I_t be the torsion subgroup of I. We then have $0 \to I_t \to I \to I/I_t \to 0$ exact, inducing

$$0 \to H^3_{\rm sym}(G, I_t) \to H^3_{\rm sym}(G, I) \to H^3_{\rm sym}(G, I/I_t).$$

The $H^3_{\rm sym}(G, I_t)$ is a torsion group since I_t is torsion and G is finite; but $H^3_{\rm sym}(G, I_t)$ is isomorphic to a subgroup of $H^3_{\rm sym}(G, I)$, the $H^3_{\rm sym}(G, I)$ has no torsion so $H^3_{\rm sym}(G, I_t) = 0$.

We conclude by showing $H^3_{\rm sym}(G, I/I_t) = 0$. To do this consider $I/I_t \to I/I_t \otimes_Z Q$, ($Q$ the rationals). This map is injective and induces

$$H^3_{\rm sym}(G, I/I_t) \to H^3_{\rm sym}(G, I/I_t \otimes Q),$$

which is also injective. But the latter group is zero since Q is a field of characteristic zero.

5. $H^3_{\rm sym}(G, G_a) = 0$

Let k be a field. Let $G = \operatorname{Spec} E$ be a finite group scheme over k (as defined at the beginning of Section 3). Let $A = \operatorname{Hom}(E, k)$, be the dual space. Then Spec A can be given the structure of a finite group scheme over k. Let G_a be the functor from affine schemes to groups defined by: $G_a(\operatorname{Spec} R) = R, +$ (the additive group of R).

Then letting G act trivially on G_a, one is in position to define the cohomology groups $H^n(G, G_a)$ as in [2], I, Section 5. The complex $C(G, G_a)$ which is defined there, now becomes: $C^n(G, G_a) = E^{\otimes n} = \mathrm{Hom}(A^{\otimes n}, k)$ and the boundary map is the same as in Section 1.

Remark. If one replaces G_a by the functor called $\mathcal{O}$ in [2] ($\mathcal{O}(\mathrm{Spec}\, R) = R$ as a ring), then $H^{\cdot}(G, G_a)$ becomes a graded ring. This ring is shown in [3] to be just the graded ring $H^{\cdot}(A, k)$, which we have essentially computed in Theorem 3.1.

One can also define $C^2_{\mathrm{sym}}(G, G_a)$ and $C^3_{\mathrm{sym}}(G, G_a)$ using the definition of $C^2(G, G_a)$ and $C^3(G, G_a)$ given in [2], I, or [3]. We do not carry this out explicitly here since this involves more category theory than seems appropriate in this paper. We simply remark that $C^2_{\mathrm{sym}}(G, G_a)$ turns out to be $E^{\otimes 2}_{\mathrm{sym}}$ and $C^3_{\mathrm{sym}}(G, G_a)$ to be $E^{\otimes 3}_{\mathrm{sym}}$. Define $H^3_{\mathrm{sym}}(G, G_a)$ to be the kernel of δ on $C^3_{\mathrm{sym}}(G, G_a)$ modulo the image of δ on $C^2_{\mathrm{sym}}(G, G_a)$. Then $H^3_{\mathrm{sym}}(G, G_a)$ is just $H^3_{\mathrm{sym}}(A, k)$; $H^3_{\mathrm{sym}}(A, k) = 0$, by Theorem 3.2, and we have proven:

THEOREM 5.1. *Let G be a finite group scheme over a field k. Let G_a be the functor from affine scheme to Abelian groups, where $G_a(\mathrm{Spec}\, R)$ equals R, $+$ (the additive group of R). Then if G acts trivially on G_a,*

$$H^3_{\mathrm{sym}}(G, G_a) = 0.$$

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